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L^2 INVARIANTS FOR FINITELY GENERATED GROUPS

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This problem list concerns the study of the asymptotic behavior of some natural invariants of finitely generated groups.

1. Approximation of L_2-torsion.

Let M be a closed Riemannian manifold and let

$$\Gamma = \pi_1(M) > \Gamma_1 > \Gamma_2 > \dots$$

be a decreasing sequence of finite index normal subgroups of Γ with $\bigcap \Gamma_n = 1$. Let $\tilde{M}$ be the universal cover of M and let $M_n = \tilde{M}/\Gamma_n$.

In addition, suppose that M is aspherical with $\dim M = 2k+1$ and $\beta_j^{(2)} = 0$ for every j.

Problem 1.1. *Do we have*

$$\rho^{(2)}(\tilde{M}) = (-1)^k \lim \frac{\log(|tors(H_k(M_n))|)}{|\Gamma : \Gamma_n|}?$$

Remark. Note that, if M is a hyperbolic 3-manifold then $\rho^{(2)}(\tilde{M}) = -\frac{1}{6\pi} vol(M)$. Moreover, if Γ has an elementary amenable normal subgroup, then Question holds for Γ.

Remark. Note that, the question is true for groups that are limits of left-orderable amenable groups. Also for groups where every non-trivial finitely generated subgroup surjects on $\mathbb{Z}$).

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Problem 1.2. *Is*

$$\rho^{(2)}(\tilde{M}) = \lim_n \frac{\rho(M_n)}{|\Gamma : \Gamma_n|}?$$

The following is equivalent to the previous problem.

Problem 1.3. *Let $A \in M_l(\mathbb{Z}\Gamma)$, denote $A_k = A/\Gamma_k$. Is it true that*

$$tr_{L\Gamma}(\log A^*A) = \lim \frac{tr(\log A_k^* A_k)}{|\Gamma : \Gamma_k|}?$$

Problem 1.4. *Is*
$$\lim \frac{b_{F_p}(\Gamma_n)}{|\Gamma : \Gamma_n|} = \lim \frac{rk(\Gamma_n)}{|\Gamma : \Gamma_n|}$$
for any (not necessarily finitely generated) group?

Let Γ be a finitely presented residually p-group. Let Γ_n be a normal p-chain with $\bigcap \Gamma_n = 1$, then
$$\lim \frac{b_{\mathbb{Q}}(\Gamma_n)}{|\Gamma : \Gamma_n|} \leq \lim \frac{b_{F_p}}{|\Gamma : \Gamma_n|} \leq \lim \frac{rk(\Gamma_n)}{|\Gamma : \Gamma_n|}$$

Problem 1.5. *Can these inequalities be strict?*

2. Orbit Equivalence of Measure Preserving Actions

Problem 2.1. *Let Γ_1 and Γ_2 be infinite countable groups. Does $\Gamma_1 \times \Gamma_2$ have fixed price 1?*

Definition: Γ is almost treable if it admits a free action on probability space such that the equivalence relation associated to this action is almost treeable, i.e., it is an increasing union of treeable subequivalence relations.

The property is stable under measure equivalence and taking subgroups. From the property it follows that Γ has the Haagerup property and is sofic. Moreover, $\lambda_{cb}(\Gamma) = 1$ if Γ is almost treeable.

Examples include: $\mathbb{F}_2 \times H$, where H is amenable.

Problem 2.2. *Are the fundamental groups of hyperbolic 3-manifolds almost treeable?*

Problem 2.3. *Are 2- and 3-generated groups topologically orbit equivalent?*

Let Γ be a non-amenable group with a probability measure preserving action of Γ on (X, μ).

Problem 2.4. *Is it true that for any N there exist measurable subsets $A_g \subseteq X$ $(g \in \Gamma)$ such that $\prod\limits_{x \in A_g} (x, xg)$ is a forest with $\sum\limits_{g \in \Gamma} \mu^2(A_g) > N$?*

3. More problems

Problem 3.1. *Do closed, aspherical manifolds with OE fundamental groups have the same simplicial volume?*

If, in addition, the simplicial volumes are positive, are the dimensions of the manifolds equal?

Problem 3.2. *Let $\Gamma/\mathbb{Z}_2$ be a sofic group. Is Γ sofic?*

It is known that for every finitely generated group we have
$$\beta_1^{(2)}(\Gamma) \leq d(\Gamma) - 1.$$
where d stands for the minimal number of generators of Γ.

A group Γ is normally generated by $S \subseteq \Gamma$ if the only normal subgroup of Γ containing S is Γ itself. Let $nrk(\Gamma)$ be the normal rank of Γ, i.e., $nrk(\Gamma)$ is the minimal number of normal generators.

Problem 3.3. *Do we have $\beta_1^{(2)}(\Gamma) \leq nrk(\Gamma) - 1$ for a torsion free group Γ?*

Remark. The question is true for groups that are limits of left-orderable amenable groups. Also for groups where every non-trivial finitely generated subgroup surjects on $\mathbb{Z}$).

4. The Atiyah Conjecture

The deep-fall property and the Atiyah Conjecture

This notion was used in the paper "Sumultiplicativity and the Hanna Neumann Conjecture". Let $\hat{Y}$ be a complex with a free action by a left-orderable group Γ and $i \geq 0$. This induces a Γ-invariant total order on the set of i-cells in $\hat{Y}$, $\Sigma_i^{\hat{Y}}$. For $\sigma \in \Sigma_i^{\hat{Y}}$ and $E \subseteq \Sigma_i^{\hat{Y}} \setminus \{\sigma\}$, let

$$[E < \sigma] := \{\tau \in E | \tau < \sigma\}.$$

We say that *σ falls into E* if $\partial\sigma \in \overline{\partial(\ell^2(E))}$. A cell $\sigma \in \Sigma_i^{\hat{Y}}$ is called **order-essential** if it falls into $[\Sigma_i^{\hat{Y}} < \sigma]$, i.e.

$$\partial\sigma \in \overline{\partial(\ell^2[\Sigma_i^{\hat{Y}} < \sigma])}.$$

Call it **order-inessential** otherwise. Let $\mathbb{E}_i^{\hat{Y}}$ and $\mathbb{I}_i^{\hat{Y}}$ denote the sets of order-essential and order-inessential edges in $\hat{Y}$, respectively. We say that the Γ-action on $\hat{Y}$ has the **deep-fall property**, or more precisely i-deep-fall property, if for any $\sigma \in \mathbb{E}_i^{\hat{Y}}$ we have $\partial\sigma \in \overline{\partial(\ell^2[\mathbb{I}_i^{\hat{Y}} < \sigma])}$.

The argument in "Sumbmultiplicativity ..." implies that each free action of a left-orderable group Γ on a complex with the deep-fall property for some i provides an instance when the (integral) Atiyah Conjecture holds. (Represent the ith boundary map by a matrix with entries in $\mathbb{Z}\Gamma$ and see that the kernel has integral dimension.) So for future investigation we propose

Problem 4.1. *Find many examples of left-orderable groups and their free actions on complexes that have the deep-fall property in some dimension i. Find free actions by left orderable groups that do not have the deep-fall property. See what can be said about non-free actions on ordered complexes. The same will formally apply to matrices with entries in the group ring over $\mathbb{C}$.*

References