[2007/11/11]

JSJ DECOMPOSITIONS

EDITED BY PANOS PAPASOGLU

Splitting theory of groups parallels decomposition theorems in 3-manifold theory. Kneser's theorem has its analog in Grushko's theorem, Haken finiteness and hierarchies gave rise to accessibility and strong accessibility in group theory while JSJ decompositions of 3-manifolds were shown recently to extend to group theory too. One should stress that these "generalizations" from 3-manifolds to groups were by no means routine and a lot of questions remain open today. The structure theory that one obtains for groups is often much more general that the 3-manifold theory and it had many striking applications (isomorphism problem for hyperbolic groups, Tarski problems etc). We refer to [**?**] and [**?**] for the basic definitions.

1. JSJ DECOMPOSITIONS

Problem 1.1. *Is there a JSJ decomposition for finitely presented groups over small groups?*

Remark. Dunwoody [**?**] has announced recently a much more general result for groups acting on $\mathbb{R}$-trees that seems to imply the existence of JSJ decompositions over small groups. Guirardel and Levitt have also proposed a generalized JSJ theory in the spirit of outer space [**?**], [**?**].

Remark. The first case to look at is the case of decompositions over solvable Baumslag-Solitar groups. If such a decomposition exists then the edge groups of the decomposition won't necessarily be small (see [**?**]). On the other hand there is a natural conjecture for the enclosing groups: one expects them to be fundamental groups of complexes of groups where the underlying complex is a square complex homeomorphic to a surface and all edges and faces are stabilized by a fixed group F (the fiber). However the homomorphisms from edge groups to face groups are not necessarily isomorphisms.

We call a splitting of G over C *unfolded* if G does not split over any proper subgroup of C.

Problem 1.2. *Let G be a finitely presented group that does not split over a virtually abelian group. Can G have infinitely many unfolded splittings over distinct subgroups isomorphic to $\mathbb{F}_2$?*

Remark. A splitting of a group G over a group C_1 is called elliptic with respect to a splitting of G over C_2 if C_1 fixes a point of the Bass-Serre tree of the splitting over C_2. Otherwise it is called hyperbolic.

A pair of splittings of G over C_1, C_2 can be elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic, hyperbolic-hyperbolic.

A crucial observation for the JSJ theory of splittings of 1-ended finitely presented groups over $\mathbb{Z}$ is that any two splittings over infinite cyclic groups C_1, C_2 are either elliptic-elliptic or hyperbolic-hyperbolic.

The Bestvina-Feighn accessibility theorem ([**?**]) gives a bound on the number of elliptic-elliptic splittings over $\mathbb{Z}$ (and more generally over small groups). So the main issue for the JSJ theory is understanding hyperbolic-hyperbolic splittings. In the case of $\mathbb{Z}$-splittings it is easy to produce infinitely many distinct hyperbolic-hyperbolic splittings, just consider the splittings corresponding to simple closed curves on a surface. The JSJ theory says that in fact this is the only way in which such splittings arise. So the problem above asks whether there are families of hyperbolic-hyperbolic splittings over the free group of rank 2, $\mathbb{F}_2$.

Problem 1.3. *Is the isomorphism problem solvable for the graphs of groups for which all edges and vertices are labelled by* $\mathbb{Z}$*?*

Remark. One of the main applications of the JSJ theory was the solution of the isomorphism problem for hyperbolic groups. In general however there is no canonical JSJ decomposition (see though [**?**]) and one does not have a good description of the set of all JSJ decompositions. The simplest case for which this question is open is that of graphs of groups where all vertex and edge groups are isomorphic to $\mathbb{Z}$. In this case an algorithmic description is equivalent to the isomorphism problem stated above. Levitt remarks that the isomorphism problem for graphs of groups where all edges and vertices are labelled by $\mathbb{Z}^4$ is unsolvable.

A Dehn twist of G is an automorphism ϕ of the following form: either G splits as $A *_C B$, $t \in C$ is central, $\phi(a) = a$ for $a \in A$ and $\phi(b) = tbt^{-1}$ for $b \in B$; or G is an HNN extension $A*_C$, with stable letter s, $t \in C$ is central, $\phi(a) = a$ for $a \in A$ and $\phi(s) = ts$.

Problem 1.4. *Let G be a* $CAT(0)$ *group with* $Out(G)$ *infinite. Is it true that G admits a Dehn twist of infinite order?*

Remark. Does G even have any nontrivial splittings?

2. Accessibility Questions

We recall the definition of strong accessibility of a group G over a family of subgroups C: decompose G as a graph of groups with edge groups in C. Then decompose the vertex groups of this graph as graphs of groups with edge groups in C and so on. If there is such a series of decompositions that terminates (i.e. at some stage the vertex groups do not admit any decomposition over C) then the group is called *strongly accessible* over C. We note that one asks only for the existence of such a series of decompositions and not that *every* such series terminates. This is analogous to hierarchies for 3-manifolds.

Problem 2.1. *Let C be the class of finite and virtually-*$\mathbb{Z}$ *groups. Are hyperbolic groups strongly accessible over C?*

Remark. Grushko's theorem implies that finitely generated groups are strongly accessible over the trivial group. Dunwoody's theorem [**?**] implies that finitely presented groups are strongly accessible over $C = \{\text{finite groups}\}$. Delzant and Potyagailo [**?**] showed that hyperbolic groups with no 2 torsion are strongly accessible over the class C of finite and 2-ended groups. So the question remaining here is whether the no-2 torsion assumption can be removed.

Problem 2.2. *Let C be the class of finite and virtually-*$\mathbb{Z}$ *groups. Are finitely presented groups strongly accessible over C?*

Remark. One can ask this more generally for splittings over small groups.

Problem 2.3. *Is there a 1-ended finitely presented group G such that $G \simeq G*_{\mathbb{Z}}$?*

One may ask more generally

Problem 2.4. *Let C be the class of finitely generated groups. Are finitely presented groups strongly accessible over C?*

Problem 2.5. *Let $G = A *_{C_n} B_n$, $n \in \mathbb{N}$ where G is a finitely generated, C_n are slender and $C_n < C_{n+1}$ for all n. Is it true that $G = A *_C B$ where $C < C_n$ for all n? Similarly for HNN extensions.*

Remark. This was shown by Rips-Sela [**?**] for C_n isomorphic to $\mathbb{Z}$. Weidmann pointed out that one may pose the question even more generally omitting 'finitely generated' for G. It is true for finitely presented groups [**?**].

3. Group splittings and asymptotic topology

One may pose a very general question: What meaningful geometric properties distinguish Cayley graphs from arbitrary metric spaces? Here geometric of course is in the sense of Gromov's quasi-isometries. A lot of the recent work in geometric group theory falls in this frame. Stallings' ends theorem, Gromov's polynomial growth theorem and Varopoulos isoperimetric inequality are three deep positive results distinguishing Cayley graphs from general metric spaces. On the other hand one may see Grigorchuk's intermediate growth examples and Rips-Sapir-Birget-Olshanskii-Bridson isoperimetric inequalities examples as negative results to the above question, saying that "everything is possible" for Cayley graphs.

Splitting theory has provided some further positive results in this quest and there is some reasonable hope for further progress. We state here some specific questions in this spirit.

Problem 3.1. *Let G be a hyperbolic group such that ∂G is not separated by a Cantor set. Is it true that ∂G is not separated by a simple path?*

Remark. Since Stallings ends theorem the theory of splittings has been related to "asymptotic topology" of Cayley graphs. The relation became clearer after Bowditch's work. In the case of hyperbolic groups the asymptotic topology of the group is reflected on the topology of the boundary which is a compact metrizable space. Bowditch [**?**] and Swarup [**?**] showed that the boundary of a one ended hyperbolic group has no cut points. Bowditch [**?**] showed that if the boundary has a local cut point then the group splits over a 2-ended group.

Problem 3.2. *Let G be a finitely generated group which does not split over a finite group. Is it true that (any subset of) a quasi-ray does not separate coarsely the Cayley graph of G?*

Remark. By quasi-ray we mean a uniform embedding of $\mathbb{R}^+$ in the Cayley graph of G. We remark that the answer to the above question is positive for finitely presented groups ([**?**], [**?**]). This question is similar to Bowditch's no cut point theorem for finitely generated groups. We recall below the relevant definitions.

A map $f : X \to Y$ between two metric spaces is called a uniform embedding if

i) there are C, D such that $d(f(x), f(y)) \leq Cd(x, y) + D$ for all $x, y \in X$

ii) if $d(x_n, y_n) \to \infty$ then $d(f(x_n), f(y_n)) \to \infty$ for any two sequences $(x_n), (y_n)$ in X.

We say that a set A is coarsely contained in a set B if A is contained in a finite neighborhood, $N_K(B)$, of B.

We say that a subset Y of X coarsely separates X if for some finite neighborhood $N_K(Y)$, $X - N_K(Y) = X_1 \sqcup X_2$ with X_1, X_2 open and neither X_1 nor X_2 is coarsely contained in Y.

It is worth remarking that the no quasi-ray separates question can be seen also as generalizing Stallings theorem. Indeed one may restate Stallings theorem as follows:

Let G be a finitely generated group that does not split over a finite group. Then a point does not separate coarsely the Cayley graph of G.

So if the answer to the above problem is positive it gives a strengthening of Stallings ends theorem. We note also that the exact analog of the no-cut point theorem would be to show that quasi-"horo-balls" do not separate, rather than quasi-rays. This is not known for finitely presented groups though, in fact it is not even clear what is the right notion of quasi-"horo-ball".

Problem 3.3. *Are one-ended CAT(0) groups semi-stable at infinity?*

Remark. The no cut point theorem for hyperbolic groups implies (via work of Bestvina-Mess [**?**]) that hyperbolic groups are semi-stable at infinity. Although it is known now [**?**] that CAT(0) boundaries have no cut points the semi-stability question is still open for CAT(0) groups.

Problem 3.4. *Are splittings of one-ended finitely generated groups over 2-ended groups invariant under quasi-isometries?*

Remark. The answer is positive for finitely presented groups [**?**]

Problem 3.5. *Let G be a finitely generated group such that there is a sequence of quasi-circles that separate its Cayley graph. Is G a virtually surface group?*

Remark. It follows by work of Bowditch [**?**] that this is true for finitely presented groups. By a sequence of quasi-circles that separate we mean the following: we take a union of circles $X = \bigcup\{C_n\}$ in $\mathbb{R}^2$ with radii tending to infinity and we consider a uniform embedding f from X to the Cayley graph of G. Now we assume that for some fixed N the N-neighborhood of $f(C_n)$ separates the Cayley graph and at least 2 components are not contained in the n neighborhood of $f(C_n)$.

For example in the hyperbolic or Euclidean plane one can take C_n to be a sequence of circles (i.e. boundaries of balls). In the case of the Euclidean plane these are quasi-isometrically embedded but this is not the case for the hyperbolic plane.

Problem 3.6. *Are splittings over virtually-$\mathbb{Z}^2$ groups invariant under quasi-isometries? More precisely let G be a one-ended finitely presented group that does not split over a 2-ended group.*

Suppose that G splits over virtually-$\mathbb{Z}^2$. Is it true a group quasi-isometric to G splits over virtually-$\mathbb{Z}^2$?

Remark. One might ask more generally whether JSJ decompositions over virtually-$\mathbb{Z}^2$ groups are invariant under quasi-isometries.

Problem 3.7. *Let G be a group with asdim $(G) \geq n$, $n > 3$. Assume that a uniformly embedded copy of $\mathbb{Z}^{n-2}$ separates coarsely the Cayley graph of G. Is it true then that G splits over virtually-$\mathbb{Z}^{n-2}$?*

Remark. Both in Stallings theorem and in Bowditch JSJ theory there are some classes of groups that can be thought of as "exceptional". In Stallings theorem these are the groups with 2 ends (all commensurable to $\mathbb{Z}$) and in Bowditch's theorem it is hyperbolic triangle groups (although their boundary has local cut points they do not split). As one tries to generalize these theorems to splittings over $\mathbb{Z}^n$ it is natural to expect that the number of "exceptions" increases. The question above avoids this technical issue by formulating the problem so that exceptions are ruled out.

Some evidence in favor of this is provided by [**?**] and [**?**].

It is interesting to note that some of the above questions can be stated also for locally connected continua (especially homogeneous continua). For example the question on quasi-rays that separate has the following twin:

Problem 3.8. *Let X be a locally connected homogeneous continuum of dimension* ≥ 2. *Is it true that X is not separated by an arc?*

Remark. The answer is positive in the case of simply connected continua ([**?**]) which heuristically corresponds to the case of finitely presented groups. Several people have noted affinities between continua and groups. It is worth noting that locally connected homogeneous continua of dimension 1 were classified by Anderson ([**?**]) while boundaries of hyperbolic groups of dimension 1 were classified by Kapovich-Kleiner ([**?**]). One does not know much in either case when dimension is greater than 1.

REFERENCES