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HYPERGRAPH TURN PROBLEM

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Here we present a selection of some open questions related to the hypergraph Turán problem.

Let $[n]$ denote the interval $\{1,\ldots,n\}$. For a set X and an integer k, let $\binom{X}{k} = \{Y \subseteq X \mid |Y| = k\}$ be the family of all k-subsets of X. By a *k-graph F* we understand a k-uniform set system, that is, F is a pair (V,E) where V is the set of vertices and $E \subseteq \binom{V}{k}$. For convenience, we will identify k-graphs with their edge set. Thus e.g. $|F| = |E(F)|$ denotes the *size* of F.

Let $\mathcal{F}$ be a family of k-graphs. A k-graph G is *$\mathcal{F}$-free* if G does not contain any member of $\mathcal{F}$ as a (not necessarily induced) subgraph. The *Turán function* is

$$\mathrm{ex}(n,\mathcal{F}) = \max\{|G| \mid G \subseteq \tbinom{[n]}{k},\ G \text{ is } \mathcal{F}\text{-free}\}.$$

It goes back to the fundamental paper of Turán [**?**]. The *Turán density* is

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{k}};$$

it not hard to show that the limit exists. If $\mathcal{F} = \{F\}$, then we abbreviate $\mathrm{ex}(n,\{F\})$ to $\mathrm{ex}(n,F)$, etc.

We refer the reader to the surveys by Füredi [**?**], Sidorenko [**?**], and Keevash [**?**].

1. Complete hypergraphs

Let $K_m^k = \binom{[m]}{k}$ be the complete k-graph on m vertices. Erdős offered a money prize for determining $\pi(K_m^k)$ for at least one pair k,m with $m > k \geq 3$; the highest money value of the prize we found in the literature is \$3000 (Frankl and Furedi[**?**, Page 323]). It is still unclaimed.

We also refer the reader to the survey by Sidorenko [**?**] that discusses $\pi(K_m^k)$ for some other k,m in addition to those discussed below.

K_4^3 minus an edge

Let K_4^- be obtained from K_4^3 by removing one edge.

Conjecture 1.4. $\pi(K_4^-) = \frac{2}{7}$

Conjecture 1.1. $\pi(K_4^3) = \frac{5}{9}$.

Remark. Fon-der-Flaass [**?**] presented a construction of K_4^3-free graphs from digraphs. A weakening of Conjecture **??** is that Fon-der-Flaass' construction cannot beat $\frac{5}{9}$; some progress in this direction was made by Razborov [**?**].

Remark. Kalai [**?**] (see also [**?**, Section 11]) presented an interesting approach to $\pi(K_4^3)$.

Intro to this problem ...

Problem 1.3. *This problem is just a placeholder. Replace it bu a real problem, or we can jsut delete it later.*

Conjecture 1.2. $\pi(K_m^3) = 1 - \left(\frac{2}{m-1}\right)^2$.

Remark. A construction that achieves the lower bound can be found in [**?**, Section 7]. Mubayi and Keevash (see [**?**, Section 9]) found a different construction (via digraphs).

Let us mention here another very interesting question for whose solution de Caen [**?**, Page 190] offered 500 Canadian dollars.

Problem 1.3. *Does* $k(1 - \pi(K_{k+1}^k))$ *tend to* ∞ *as* $k \to \infty$*?*

2. TURN FUNCTIONS FOR BOOKS

Let *the book* $B_{k,m}$ consist of m edges sharing $k - 1$ common points plus one more edge that contains the remaining m points and is disjoint otherwise. Let us exclude the case $m \le 1$ when $\pi(B_{k,m}) = 0$. The hypergraph problems for books turned out (relatively) more tractable. We know $\mathrm{ex}(n, B_{k,m})$ exactly for all large n, when $2 \le m \le k \le 4$, see [¡a href=”” class=”cite”¿bollobas:74,frankl+furedi:83,fur
Frankl and Füredi [**?**] determined $\pi(B_{k,2})$ for $k = 5, 6$; in both cases the lower bounds comes by blowing up a small design. The following question is still open (see Frankl and Füredi [**?**, Conjecture 1.5]):

Problem 2.1. *Determine* $\mathrm{ex}(n, B_{5,2})$ *and* $\mathrm{ex}(n, B_{6,2})$ *exactly for all large n.*

Remark. One difficulty for this problem is that it is not clear how to prove *the stability property*, that is, that all almost extremal graphs have similar structure.

Conjecture 2.2.

$$\pi(B_{5,5}) = \frac{40}{81},$$

and

$$\pi(B_{6,6}) = \frac{1}{2}.$$

Remark. The lower bounds come from a “bipartite” construction. It was proved in [**?**] that $\pi(B_{5,5}) \le 0.534...$ and that the bipartite construction is not optimal for $\pi(B_{k,k})$ when $k \ge 7$.

Remark. The Turán density is unknown for $B_{5,3}$ and $B_{5,4}$ which is an interesting (and perhaps tractable) open problem.

3. TIGHT 5-CYCLE

Note to the editors: The items in this section are in the wrong order. The next version of the code will provide an easy way to reorder the problems in a section.

Mubayi and Rödl [?] have given bounds on $\pi(C_5^3)$, where C_5^3 is the tight 3-graph 5-cycle:

$$C_5^3 = \{123, 234, 345, 451, 512\}.$$

In particular, the lower bound $\pi(C_5^3) \geq 2\sqrt{3} - 3$ comes from the following construction: partition the vertex set into two parts A and B, take all triples that intersect A precisely in 2 vertices, and recursively repeat this construction within B. Finding the optimal ratio between $|A|$ and $|B|$ gives the required. Razborov's [?] flag algebra computations showed that $\pi(C_5^3) < 0.4683$ (note that $2\sqrt{3} - 3 = 0.4641...$). This makes the following conjecture plausible.

Conjecture 3.1. $\pi(C_5^3) = 2\sqrt{3} - 3$

Tight 5*-Cycle Minus an Edge*

Let the 3-graph C_5^- be obtained from C_5^3 by removing one edge. An example of a C_5^--free 3-graph can be obtained by taking a complete 3-partite 3-graph and repeating this construction recursively within each of the three parts. This gives density $1/4$ in the limit.

Conjecture 3.2. $\pi(C_5^-) = 1/4$.

4. Ruzsa-Szemerédi Theorem and Relatives

Let the 3-graph C_5^- be obtained from C_5^3 by removing one edge. An example of a C_5^--free 3-graph can be obtained by taking a complete 3-partite 3-graph and repeating this construction recursively within each of the three parts. This gives density $1/4$ in the limit.

Conjecture 4.1. *For any $r \geq 3$ and $s \geq 4$ we have*

$$f^r(n, s(r-2) + 3, s) = o(n^2).$$

Remark. In [?] it is proved that $f^r(n, s(r-2) + [\log_2 s], s) = o(n^2)$. The first remaining open case is to prove the conjecture for $f^3(n, 7, 4)$ (probably very hard).

Remark. One possible direction here is to look at multiple hypergraphs (when the same r-tuple can appear a multiple number of times) and ask for $F^r(n, p, s)$ maximum size of an r-multi-hypergraph such that every s-set spans at most p edges. See Füredi and Kündgen [?] for results in the graph case ($r = 2$).

A related question is as follows. Let A, B, and C be disjoint sets each of size n. Let $M_1, \ldots, M_l$, be matchings, where each edge of M_i has one point in each of A, B, and C. The forbidden configuration is: three edges abc, $a'b'c'$, $a''b''c''$ all in some M_i and one edge of the form $ab'c''$ in some other M_j (that is, the edge from M_j crosses the three edges of M_i). Additionally, we require that the union of all matchings M_i makes a simple (linear) 3-graph, call it M.

Conjecture 4.2.

$$|M| = o(n^2).$$

The following conjecture seems to be related.

Conjecture 4.3. *Let F be a graph and $\alpha > 1$ be such that $\mathrm{ex}(n, F) = \Omega(n^{\alpha})$. Then for any $\epsilon > 0$ there is n_0 so that if $n > n_0$ and a graph H is the edge-disjoint union of $m = \lceil \epsilon n^{\alpha} \rceil$ copies of F, then H contains another copy of F (i.e. has at least $m + 1$ copies of F).*

REFERENCES