

Problem .1. Let

$$Q_n^{(\alpha, \beta)} := \sum_{k=0}^n \binom{n}{k} f_k f_{n-k} ((x + \alpha)_k (x + \beta)_{n-k} - (x + \alpha + \beta)_k (x)_{n-k}),$$

where $\alpha, \beta > 0$, $f_k^2 \geq f_{k-1} f_{k+1}$, and $(x)_k = x(x+1) \cdots (x+k-1)$. Then the Maclaurin coefficients of $Q^{(\alpha, \beta)}$ are non-negative.

Remark. [D. Karp] I have proved the case $\alpha = \beta = 1$ and formulated a number of generalizations of this conjecture in <http://arxiv.org/abs/1203.1482>

The following conjecture of (McNamara et al.) was proved by P. Brändén. Let $p(x) := \sum_{k=0}^n a_k x^k$, with $a_{-1} := a_{n+1} := 0$. If $p \in \mathcal{L}\text{-}\mathcal{P}^+$, then

$$\sum_{k=0}^n \begin{vmatrix} a_k & a_{k+1} \\ a_{k-1} & a_k \end{vmatrix} x^k \in \mathcal{L}\text{-}\mathcal{P}.$$

S. Fisk generalized this statement to polynomials with coefficients formed from the determinants of 3×3 matrices in the conjecture below.

Conjecture .2. If $p \in \mathcal{L}\text{-}\mathcal{P}^+$, then

$$\sum_{k=0}^n |ccc a_k a_{k+1} a_{k+2} a_{k-1} a_k a_{k+1} a_{k-2} a_{k-1} a_k|$$

$x^k \in \mathcal{L}\text{-}\mathcal{P}^+$.

Problem .3. Let $\{\phi_k(x)\}_{k=0}^{\infty}$ and $\{f_k\}_{k=0}^{\infty}$ be log-concave. When is $\sum f_k \phi_k(x)$ log-concave?

4. COMBINATORICS AND GRAPH THEORY

Let S_{2n} be the convex hull of all adjacency matrices of perfect matchings. Let

$$\mu_{2n} := \min\{\text{haf}(B) : B \in S_{2n}\},$$

where

$$\text{haf}(B) := \frac{1}{n! 2^n} \frac{\partial}{\partial x_1 \cdots \partial x_{2n}} (x^T B x)^n$$

is the *hafnian* of B .

Problem .1.

(1) Is it true that

$$\alpha := \liminf_n \frac{\log \mu_{2n}}{2n} > -\infty ?$$

(2) Estimate μ_{2n} .

The type- D_n Eulerian polynomials are given by

$$E(x) := \sum_w x^{\#\{s: \text{length}(ws) < \text{length}(w)\}}.$$

Problem .2. *Prove that $E(x)$ has only real zeros.*

Problem .3. *Is there a decent multivariate extension of the Chudnovsky-Seymour Theorem that the vertex independent set polynomial of a claw-free graph has only real roots?*

Problem .4. *Generalize the Chudnovsky-Seymour (Heilmann-Lieb) Theorem. Extend the common interlacing property to several variables.*

5. ZEROS OF DERIVATIVES

Problem .1. *Let $p \in \mathbb{R}[x]$, $\deg(p) = n > 2$. There are at least $n - 1$ points of extreme curvature $\kappa(p)$ (more generally where $\kappa'(p) = 0$) [?].*

Problem .2. *What are the zero spacing effects of higher order differential operators on functions which belong to $\mathcal{L}\text{-}\mathcal{P}$?*

6. ORTHOGONAL POLYNOMIALS

Let (L_n^α) denote the generalized Laguerre polynomials. It is known that L_m^α and L_n^α can have common zeros if $|m - n| > 1$. Is the number of common zeros linked to arithmetic properties of α ?

Conjecture .1. *There are no common zeros between L_m^α and L_n^α , $|m - n| > 1$, for α rational.*

Problem .2. *Let $P_n^{\alpha,\beta}$ denote the Jacobi polynomial of deg n with parameters $\alpha, \beta > -1$. Then $P_n^{\alpha',\beta'}$ and $P_n^{\alpha,\beta}$ interlace for $\alpha \approx \alpha'$, and $\beta \approx \beta'$. Does the failure of interlacing correspond to some physical law being “violated”?*

Problem .3. *Consider a subspace S of $\mathbb{C}[x]$ of codimension k (defined by k linear constraints, explicitly given). Given a scalar product on $\mathbb{C}[x]$, find an orthogonal basis of S (ordered by degree); some degrees will be missing.*

- (1) *Which degrees get skipped?*
- (2) *Characterize these orthogonal polynomials.*
- (3) *Does this have anything to do with lacunary Padé approximants?*

7. RIEMANN Ξ -FUNCTION

Let

$$\Xi(z) := \int_{-\infty}^{\infty} \Phi(t) \cos(zt) dt,$$

where

$$\Phi(t) := \sum_{n=1}^{\infty} (2n^4 \pi^2 e^{9t} - 3n^2 \pi e^{5t}) \exp(-n^2 \pi e^{4t}).$$

Let $p_k(z)$ be the polynomials orthogonal with respect to Φ on $(-\infty, \infty)$. Define

$$P_n(z) := \frac{\det [p_i(z_j)]_{i,j=1}^n}{\det [z_i^{j-1}]_{i,j=1}^n}, \quad \text{and} \quad Q_n(z) := \frac{\det [p_i(z_j)]_{i,j=2}^n}{\det [z_i^{j-2}]_{i,j=2}^n}.$$

Problem .1. Investigate the zeros of P_n and Q_n .

Remark. [D. Dimitrov] The following generalization of the result mentioned above is true.

Let μ be an even, positive measure on \mathbb{R} , and $F(z) := \int_{-\infty}^{\infty} \cos(zt) d\mu(t)$. Let $\{p_k(z)\}_{k=0}^{\infty}$ be the orthogonal polynomials with respect to μ on $(-\infty, \infty)$. Then $F \in \mathcal{L}\text{-}\mathcal{P}$ if and only if the associated polynomial $P_n[p_1, \dots, p_n]$ has purely imaginary zeros, if and only if $Q_n[p_1, \dots, p_n]$ has no purely imaginary zeros. (for all n)

Problem 8.2. Are there any non-real zeros of

$$\int_0^{\infty} \Phi^{\alpha}(t) \cos(zt) dt \quad \text{for} \quad \alpha > 0?$$

8. ZEROS OF SPECIFIC FUNCTIONS

Conjecture .1. Let

$$p_n(z) = \sum_{k=0}^n (-1)^k (2k+1) z^{k(k+1)}$$

where, $n = 2, 3, 4, \dots$. Each p_n has no real zeros for even n , and exactly two real zeros for odd n .

Problem .2. Consider the function

$$f(z) := \sum_k \frac{z^k}{(a^k + 1)(a^{k-1} + 1) \cdots (a + 1)}, \quad a > 1.$$

For which values of a is f in $\mathcal{L}\text{-}\mathcal{P}$?

Let

$$P_n(z, w) = \sum_{k=0}^n \binom{n}{k} z^k w^{k(n-k)}$$

Conjecture .3. For all $|w| > 1$, all the zeros of $P_n(\cdot, w)$ are simple and separated in modulus (by a factor of at least $|w|^2$).

Problem .4. Generalize conjecture .3 to $Q_n(z, w) = \sum_{k=0}^n a_k z^k w^{k(n-k)}$. For what class of $\{a_n\}$ does the conjecture still hold?

9. MISCELLANEOUS PROBLEMS

Problem .1. Let $K \subset \mathbb{C}$ be compact, let $\mathbb{C} \setminus K$ be connected, and let f be a continuous function on K which is holomorphic and non-zero on the interior of K . Is there a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ with no zeros in K such that $p_n \rightarrow f$?

Conjecture .2. Let $K \subset \mathbb{C}$ be compact, $\mathbb{C} \setminus K$ connected, and f be a continuous function on K which is holomorphic and non-zero on the interior of K . Suppose $K \subseteq \{z : \frac{1}{2} < \operatorname{Re} z < 1\}$, then

$$\bar{d}(\{t \in \mathbb{R} : \|\zeta(\cdot + it) - f(\cdot)\|_{\infty, K} < \epsilon\}) > 0,$$

where

$$\bar{d}(E) := \limsup_{T \rightarrow \infty} \frac{m(E \cap [0, T])}{T}.$$

Remark. [P. Gauthier] Andersson has shown that a positive answer to Problem .1 and a confirmation of Conjecture .2 are equivalent.

Problem .3. Let μ be a probability measure on $\{0, 1\}^n$. Consider

$$f(\vec{z}) = \int \vec{z}^{\vec{\alpha}} d\mu(\vec{\alpha})$$

and

$$g(\vec{\lambda}) = \int e^{\vec{\lambda} \cdot \vec{\alpha}} d\mu(\vec{\alpha}).$$

What property of g is equivalent to the stability of f ?

Problem .4. Let $p(z) \in \mathbb{C}[z]$, $\deg(p) = n$. Suppose $p(z) \neq 0$ for all $|z| \leq 1$. Define $p^*(z) := z^n \overline{p(\frac{1}{\bar{z}})}$. Then it is known that

$$p(A)p(A)^\dagger \geq p^*(A)(p^*(A))^\dagger \text{ for any contractive matrix } A.$$

(A is contractive means $\sup_x \frac{\|Ax\|_2}{\|x\|_2} < 1$.) In 2D, we replace A with a pair of commuting contractions and p with a bivariate polynomial with no zeros in $\{z : |z| \leq 1\} \times \{w : |w| \leq 1\}$. The 3D generalization fails – when does it hold?

Problem .5. Find more examples and attempt to characterize Markov processes preserving stability. Explicitly, let $\psi : \mathbb{R}_{MA}[x, y, z] \rightarrow \mathbb{R}_{MA}[x, y, z]$ be a linear operator. When is $e^{t\psi} : \mathbb{R}[x, y, z] \rightarrow \mathbb{R}[x, y, z]$ stability preserving for all $t \geq 0$?