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COMPONENTS OF HILBERT SCHEMES

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Here we list the open problems raised/discussed during the AIM workshop *Components of Hilbert Schemes*, July 19 to 23, 2010, organized by Robin Hartshorne, Diane Maclagan, and Gregory G. Smith.

1. HILBERT SCHEMES OF POINTS

Denote by $\mathrm{Hilb}^d(\mathbb{A}^n)$ the Hilbert scheme of d points in the affine space $\mathbb{A}^n$. By component we always mean an irreducible component.

Problem 1.05. *Describe the singularities of the smoothable component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*. To be more specific:*

How large can the dimension of the Zariski tangent space to this component get?

Does the maximum occur in the intersection of components? Does it occur in the smoothable component?

Problem 1.1. *Can you describe the Zariski tangent space to the smoothable component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 1.15. *What is the smallest d such that* $\mathrm{Hilb}^d(\mathbb{A}^3)$ *is reducible?*

Remark. We know that $10 < d \leq 78$. The bound $d > 10$ needs the assumption that char=0 and follows from a recent paper of Sivic.

Problem 1.2. *(1) Is there a component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$ *that exists only in characteristic p for some p?*

(2) Same question for the Hilbert schemes of curves in $\mathbb{P}^3$.

Problem 1.25. *Is there a nonreduced component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*? If so, find it.*

Problem 1.3. *(1) Give an explicit example (or show it does not happen) of a geometrically irreducible component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*, which is not fixed under the action of* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

(2) Same question for the Hilbert schemes of curves in $\mathbb{P}^3$.

Problem 1.35. *Is the Gröbner fan a discrete invariant that distinguishes the components of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 1.4. *Does there exist a nonrational component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 1.45. *If a component of the Hilbert scheme contains a smooth Borel fixed point, does the component have to be rational?*

Remark. If the ideal is a segment ideal (i.e. a monomial ideal generated in some degree s, where s is not larger than its regularity, by the maximal monomials with respect to some term ordering), then the component is known to be rational, c.f. [P. Lella, M. Roggero, *Rational components of Hilbert schemes*, [**?**]]. Note that every segment ideal is a Borel fixed ideal, but the converse is not true.

Let C be a curve.

Problem 1.5. *Can* $\mathrm{Hilb}^d(C)$ *have a component of dimension less than d?*

Remark. There exists a non-smoothable component of $dim = d > 1$.

Problem 1.55. *Give a geometric algebraic description of generic points of irreducible components of* $\mathrm{Hilb}^d(\mathbb{A}^n)$.

Problem 1.6. *Is* $\mathrm{Hilb}^8(\mathbb{A}^4)$ *reduced? More generally, develop techniques to prove the reducedness of Hilbert schemes.*

Problem 1.65. *Is it possible to define analogues of the Nakajima operators for the cohomology of a desingularization of the smoothable component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$*?*

Problem 1.7. *Does the set of monomial ideals contained in a component of* $\mathrm{Hilb}^d(\mathbb{A}^n)$ *determine that component? (Or more generally for* $\mathrm{Hilb}^P(\mathbb{P}^n)$ *where the Hilbert polynomial P is arbitrary.)*

Remark. For the analogous question for the Hilbert scheme of curves, Richard Liebling might have given a counterexample in his thesis.

2. Hilbert schemes of curves

Denote by $\mathrm{Hilb}_{d,g}(\mathbb{P}^r)$ the Hilbert scheme of curves of degree d and genus g in the projective space $\mathbb{P}^r$. By "component" we always mean an irreducible component.

Fix $d, g, r, e > 0$. Let $\mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r)$ be the open subscheme of the Hilbert scheme $\mathrm{Hilb}_{d,g}(\mathbb{P}^r)$ that parameterizes smooth curves. For each point $[C] \in \mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r)$, we define the Gauss map $C \to \mathrm{Gr}(1, r)$ sending a point of C to the tangent line at that point. Define

$$Z^e := \{[C] \in \mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r) | \text{ the Gauss map of the curve } C \text{ is inseparable of degree } p^e\}.$$

Then $\bigcup_{e\geq 0} Z^e = \mathrm{Hilb}^{sm}_{d,g}(\mathbb{P}^r)$.

Problem 2.05. *What can we say about the set Z^e? Can we construct exotic components (i.e. components that only exist in characteristic p) using this stratification? Study the action by the Galois group* $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

Remark. There are some other stratifications one may consider, e.g. the one obtained by the topological type of the ramification divisor.

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Let C_t be a family of curves in $\mathbb{P}^3$ such that a general curve in this family is a smooth complete intersection, and the special curve C_0 is smooth.

Problem 2.15. *Is C_0 always a complete intersection, assuming that the characteristic is 0?*

Remark. If the characteristic is $p > 0$, the answer is negative; if $n > 3$, the answer is negative; if C_0 is not smooth, the answer is negative. The reference is [**?**][P. Ellia, R. Hartshorne, *Smooth specializations of space curves: questions and examples*, Commutative algebra and algebraic geometry (Ferrara), 53–79, Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999].

Let $S \subset \mathbb{P}^3$ be an integral surface of degree d with a double curve D of degree e, and triple points, pinch point, etc.

Conjecture 2.2. *There exists $n_0 \in \mathbb{Z}$ such that for any smooth curve $C \subset S$ of degree $n \geq n_0$, let H_C be the irreducible component of the Hilbert scheme containing C, then a general $C' \in H_C$ is also contained in a surface $S' \subset \mathbb{P}^3$ of degree d and a double curve D' of degree e.*

Remark. The conjecture is posed in [**?**][P. Ellia, R. Hartshorne, *Smooth specializations of space curves: questions and examples*, Commutative algebra and algebraic geometry (Ferrara), 53–79, Lecture Notes in Pure and Appl. Math., 206, Dekker, New York, 1999]. It is proved to be true for $d = 3$ in [**?**][J. Brevik, F. Mordasini, *Curves on a ruled cubic surface*, Collect. Math. 54 (2003), no. 3, 269–281].

Problem 1.25. *Fix d, g, n. Find a good/sharp lower bound for the dimension of components of* $\mathrm{Hilb}_{d,g}(\mathbb{P}^n)$.

Remark. In the range $d^3 \leq \lambda(n)g^2$ there is a better bound, c.f. [**?**][D. Chen, *On the dimension of the Hilbert scheme of curves*, Math. Res. Lett. 16 (2009), no. 6, 941–954. Theorem 1.3].

Problem 2.3. *(1) Let C be of bidegree* $(3, 7)$ *on a nonsingular quadric surface in $\mathbb{P}^3$. Can C be connected to an extremal curve in* $\mathrm{Hilb}_{10,12}(\mathbb{P}^3)$*?*

(2) Given 4 *skew lines C_1 on a nonsingular quadric Q_1 in $\mathbb{P}^3$. Does there exists a family $Q_t \rightsquigarrow 2H$, and a family $C_t \subset Q_t$ such that $C_0 \subset Q_0 = 2H$ is locally Cohen-Macaulay?*

Remark. (2) implies (1).

Fix d, g. Let $H \subset \mathrm{Hilb}_{d,g}(\mathbb{P}^3)$ be an irreducible component.

Problem 2.35. *What is the largest number of points in general position you can make these curves pass through?*

Remark. This is a hard problem. If the curves are arithmetically Cohen-Macaulay, it should be treatable.

Problem 2.1. *Is the Hilbert scheme of local Cohen-Macaulay curves in $\mathbb{P}^3$ connected?*

Hartshorne-Rao modules

The following three problems are related to the Hartshorne-Rao modules. Let $\mathbb{P}^3$ be a projective space with homogeneous coordinate ring $R = k[X, Y, Z, T]$. Given a space curve C with ideal sheaf $\mathcal{I}_C$, the Hartshorne-Rao module is defined as the R-module $M_C = \oplus_{i\in\mathbb{Z}} H^1\mathcal{I}_C(i)$. A program to study the Hilbert scheme $H_{d,g}$ of space curves of degree d and genus g is outlined in [**?**][M. Martin-Deschamps, D. Perrin, Sur la classification des courbes gauches. Astrisque No. 184-185 (1990), 208 pp.]. Namely, $H_{d,g}$ can be stratified into $H_{\gamma,\rho}$ consisting of curves C such that, for every $i \in \mathbb{Z}$, $H^0\mathcal{I}_C(i)$ and $H^1\mathcal{I}_C(i)$ have fixed dimensions $\gamma(i)$ and $\rho(i)$, respectively. Now let E_ρ be the moduli space of finite length graded modules with Hilbert function ρ. There is a natural map from $H_{\gamma,\rho}$ to E_ρ, and properties of $H_{d,g}$ can be extracted from properties of E_ρ.

Problem 2.45. *Let $\rho = (p, q, r, 0, 0, \dots)$ and define $E_{p,q,r} := E_\rho$. For which p, q, r is $E_{p,q,r}$ irreducible?*

Remark. It is proved that if $4q < \max(6p + r, 6r + p)$ then $E_{p,q,r}$ is reducible ([**?**] [M. Martin-Deschamps, D. Perrin, *Courbes gauches et modules de Rao*, J. Reine Angew. Math. 439 (1993), 103–145. page 119, Theorem 2.1]). Conjecturally, if $4q \geq \max(6p + r, 6r + p)$ then $E_{p,q,r}$ is irreducible.

Problem 2.5. *Describe the irreducible component of E_ρ.*

Problem 2.55. *What do properties of the Rao modules imply about C? For example, if M_C is Gorenstein or annihilated by a linear form, does C have any nice properties?*

Remark. We might have to require C to be minimal.

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Problem 2.4. *What is the best pair (d, g) (in the vague sense that d is as small as possible and g is close to 0 as possible) such that there is a component of $\mathrm{Hilb}_{d,g}(\mathbb{P}^3)$ whose general member has an embedded point.*

Remark. An example of a pair satisfying the above condition is $d = 4$, $g = -15$, c.f. [D. Chen, S. Nollet, *Detaching embedded points*, [**?**]].

3. Other problems

Problem 3.1. *What is the geography of locally Cohen-Macaulay surfaces in $\mathbb{P}^4$? To be more precise, what numerical invariants (degree, sectional genus, etc.) occur?*

The following two problems are about multigraded Hilbert schemes, which parametrize all ideals in a polynomial ring which are homogeneous and have a xed Hilbert function with respect to a grading by an abelian group [**?**] [Haiman and Sturmfels, Multigraded Hilbert schemes. J. Algebraic Geom. 13 (2004), no. 4, 725769].

Problem 3.2. *Is there a multigraded Hilbert scheme with a connected component isomorphic to a fat point?*

An old question of Joe Harris:

Problem 3.4. *Does there exist a nondegenerate rigid curve in $\mathbb{P}^n$ other than the rational normal curve?*

Problem 3.3. *Is every multigraded Hilbert scheme connected if the polynomial ring is $R = k[x_1, x_2, x_3]$?*

Remark. For $R = k[x_1, \ldots, x_{26}]$, there are non-connected examples, c.f. [Francisco Santos, *Non-connected toric Hilbert-schemes*, [**?**], [**?**]]. Find examples with fewer variables.

Problem 3.5. *Are there necessary or sufficient conditions on Borel fixed monomial ideals such that they are the generic initial ideals of local Cohen-Macaulay curves?*

Fix a Hilbert polynomial P, consider the moduli space $BrV_P\mathbb{P}^n$ of branchvarieties that are equidimensional and connected in codimension 1.

Problem 3.6. *Is $BrV_P(\mathbb{P}^n)$ connected when non-empty?*

Problem 3.7. *Is there a rigid local Artinian algebra besides k^n?*

Consider $1, 4, 10, a$ where $6 \leq a \leq 10$. The general Artinian algebra with this Hilbert function is nonsmoothable.

Problem 3.8. *What is the generic point of the component it lies on?*

REFERENCES